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Mean value theorems for double zeta-functions

名古屋大学・多元数理科学研究科 松本 耕二
Kohji Matsumoto
Graduate School of Mathematics, Nagoya University
首都大学東京・理工学研究科 津村 博文
Hirofumi Tsumura
Department of Mathematics and Information Sciences
Tokyo Metropolitan University

1. Introduction and the statement of main results

The Riemann zeta-function is defined by

$$\zeta(s)=\sum_{n=1}^{\infty}n^{-s}\quad(s\in\mathbb{C};\ \Re s>1)$$

which can be continued meromorphically to the whole complex plane with a simple pole at $s=1$, and has the following functional equation:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

It is known that any non-trivial zero lies in the open strip $\{s\in\mathbb{C}\mid 0<\Re s<1\}$ ("the critical strip"), and the behaviour of $\zeta(s)$ in this strip is very important in number theory. Moreover, in view of the functional equation, in some sense it is enough to study the behaviour of $\zeta(s)$ in the right-half of this strip, that is $\{s\in\mathbb{C}\mid 1/2\leq\Re s<1\}$. Numerous researches have been done on the behaviour of $\zeta(s)$ in this region. Among them, one of the most famous theorems is as follows:

The mean value theorem for $\zeta(s)$ For any $T\geq 2$, we have

$$\int_{2}^{T}|\zeta(\sigma+it)|^{2}dt=\zeta(2\sigma)T+(2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma}+(\text{Error term})\tag{1.1}$$

for $1/2<\sigma<1$, and

$$\int_{2}^{T}|\zeta(\frac{1}{2}+it)|^{2}dt=T\log T+(2\gamma-1-\log 2\pi)T+(\text{Error term}),\tag{1.2}$$

where γ is Euler's constant.

Formula (1.2) with the error term $O(T^{3/4+\varepsilon})$ is due to Littlewood. Theorem 7.4 of Titchmarsh [12] gives a proof with the improved error term $O(T^{1/2+\varepsilon})$, and this error estimate has further been improved by many people included Balasubramanian, Huxley and so on. Formula (1.1) was first obtained by Ingham [3] and later improved by the first author, Meurman, etc.

These mean value formulas play a fundamental technical role in the analytic theory of $\zeta(s)$. Moreover, these formulas themselves suggest the following two important observations.

(a) First, it is trivial that $\zeta(\sigma+it)$ is bounded with respect to t in the region of absolute convergence $\sigma>1$, but (1.1) and (1.2) suggest that $\zeta(\sigma+it)$ seems not so large in the strip $1/2\le\sigma\le 1$, too. In fact, the well-known Lindelöf hypothesis predicts that

$$\zeta(\sigma+it)=O\left(t^{\varepsilon}\right)\qquad\left(\frac{1}{2}\le\sigma<1\right)\tag{1.3}$$

for any $\varepsilon>0$. (For $\sigma=1$, even a stronger estimate has already been known.) Formulas (1.1) and (1.2) support this hypothesis.

(b) The second observation is that the coefficient $\zeta(2\sigma)$ on the right-hand side of (1.1) tends to infinity as $\sigma\to 1/2$, hence the form of the formula should be changed at $\sigma=1/2$, which is in fact embodied by (1.2). This phenomenon suggests that the line $\sigma=1/2$ is "critical" in the theory of $\zeta(s)$. In fact, the special feature of this "critical line" (especially in connection with the Riemann hypothesis) is well-known.

As a double series analogue of $\zeta(s)$, the Euler double zeta-function is defined by

$$\zeta_2(s_1,s_2)=\sum_{m=1}^{\infty}\frac{1}{m^{s_1}}\sum_{n=1}^{\infty}\frac{1}{(m+n)^{s_2}}=\sum_{k=2}^{\infty}\left(\sum_{m=1}^{k-1}\frac{1}{m^{s_1}}\right)\frac{1}{k^{s_2}}\tag{1.4}$$

which is absolutely convergent for $s_1,s_2\in\mathbb{C}$ with $\Re s_2>1$ and $\Re(s_1+s_2)>2$ (Theorem 3 in [8]), and can be continued meromorphically to $\mathbb{C}^2$. The singularities are $s_2=1$ and $s_1+s_2=2,1,0,-2,-4,\ldots$ (Theorem 1 in [1]). Euler himself considered the behaviour of this function when s_1,s_2 are positive integers. In fact, its values at positive integers are often called the double zeta values or the Euler-Zagier double sums. It was Atkinson [2] who first studied (1.4) from the analytic viewpoint, and he proved the analytic continuation of it. As for the recent studies on the analytic side of (1.4), for example, upper-bound estimates were discussed in [4], [5], [6], and functional equations were discovered in [9], [7].

In this note we announce certain mean square formulas for (1.4), with a brief sketch of the proof. For the details, see [10].

Let

$$\zeta_2^{[2]}(s_1,s_2)=\sum_{k=2}^{\infty}\left|\sum_{m=1}^{k-1}\frac{1}{m^{s_1}}\right|^2\frac{1}{k^{s_2}}.\tag{1.5}$$

Since the inner sum is $O(1)$ (if $\Re s_1>1$), $O(\log k)$ (if $\Re s_1=1$), or $O(k^{1-\Re s_1})$ (if $\Re s_1<1$), the series (1.5) is convergent when $\Re s_1\ge 1$ and $\Re s_2>1$, or when $\Re s_1<1$ and $2\Re s_1+\Re s_2>3$.

Hereafter we write s_0 and s instead of s_1 and s_2, respectively, and consider the mean square with respect to s, while s_0 is to be fixed.

Theorem 1. *For $s_0=\sigma_0+it_0\in\mathbb{C}$ with $\sigma_0>1$ and $s=\sigma+it\in\mathbb{C}$ with $\sigma>1$, $t\ge 2$, we have*

$$\int_2^T|\zeta_2(s_0,s)|^2dt=\zeta_2^{[2]}(s_0,2\sigma)T+O(1)\qquad(T\to\infty).\tag{1.6}$$

Theorem 2. *For $s_0=\sigma_0+it_0\in\mathbb{C}$ with $\sigma_0>1$ and $s=\sigma+it\in\mathbb{C}$ with $\frac{1}{2}<\sigma\le 1$, $t\ge 2$ and $\sigma_0+\sigma>2$, we have*

$$\int_2^T |\zeta_2(s_0,s)|^2 dt=\zeta_2^{[2]}(s_0,2\sigma)T+O\left(T^{2-2\sigma}\log T\right)+O\left(T^{1/2}\right). \tag{1.7}$$

The most important, and technically the most difficult, result is the following theorem which describes the situation under the condition $\frac{3}{2}<\sigma_0+\sigma\le 2$.

Theorem 3. *Let $s_0=\sigma_0+it_0\in\mathbb{C}$ with $\frac{1}{2}<\sigma_0<\frac{3}{2}$ and $s=\sigma+it\in\mathbb{C}$ with $\frac{1}{2}<\sigma\le 1$, $t\ge 2$ and $\frac{3}{2}<\sigma_0+\sigma\le 2$. Assume that when t moves from 2 to T, the point (s_0,s) does not encounter the hyperplane $s_0+s=2$ (which is a singular locus of ζ_2). Then*

$$\int_2^T |\zeta_2(s_0,s)|^2 dt=\zeta_2^{[2]}(s_0,2\sigma)T$$
$$+\begin{cases} O\left(T^{4-2\sigma_0-2\sigma}\log T\right)+O\left(T^{1/2}\right) & (\frac{1}{2}<\sigma_0<1, \frac{1}{2}<\sigma<1)\\ O\left(T^{2-2\sigma_0}(\log T)^2\right)+O\left(T^{1/2}\right) & (\frac{1}{2}<\sigma_0<1, \sigma=1)\\ O\left(T^{2-2\sigma}(\log T)^3\right)+O\left(T^{1/2}\right) & (\sigma_0=1, \frac{1}{2}<\sigma<1)\\ O\left(T^{1/2}\right) & (\sigma_0=1,\sigma=1)\\ O\left(T^{2-2\sigma}\log T\right)+O\left(T^{1/2}\right) & (1<\sigma_0<\frac{3}{2}, \frac{1}{2}<\sigma<1). \end{cases} \tag{1.8}$$

Remark 4. In Theorems 2 and 3, the error terms $O(T^{1/2})$ are coming from the simple application of the Cauchy-Schwarz inequality. It is plausible to expect that we can reduce these error terms by more elaborate analysis.

Our theorems in this note may be regarded as double analogues of (1.1). Since the coefficient $\zeta_2^{[2]}(s_0,2\sigma)$ tends to infinity as $\sigma_0+\sigma\to 3/2$, it is natural to raise, analogously to the above (a) and (b), the following two conjectures:

(i) (a double analogue of the Lindelöf hypothesis) For any $\varepsilon>0$,

$$\zeta_2(s_0,s)=O\left(t^{\varepsilon}\right) \tag{1.9}$$

when (s_0,s) (which is not in the domain of absolute convergence) satisfies $\sigma_0>1/2$, $\sigma>1/2$, $t\ge 2$, $\sigma_0+\sigma\ge 3/2$ and $s_0+s\ne 2$;

(ii) (the criticality of $\sigma_0+\sigma=3/2$) When $\sigma_0+\sigma=3/2$, the form of the main term of the mean square formula would not be CT (with a constant C; most probably, some log-factor would appear).

Remark 5. It is not easy to find the "correct" double analogue of the Lindelöf hypothesis. Nakamura and Pańkowski [11] raised the conjecture

$$\zeta_2\left(1/2+it,1/2+it\right)=O\left(t^{\varepsilon}\right) \tag{1.10}$$

(actually they stated their conjecture for more general multiple case), and gave a certain result (their Proposition 6.3) which supports the conjecture. However, the value $\zeta_2\left(1/2+it_1,1/2+it_2\right)$ is, if $t_1\ne t_2$, not always small. In fact, Corollary 1 of Kiuchi, Tanigawa and Zhai [6] describes the situation when $\zeta_2(s_1,s_2)$ is not small. For example, if $t_2\ll t_1^{1/6-\varepsilon}$, then

$$\zeta_2\left(1/2+it_1,1/2+it_2\right)=\Omega\left(t_1^{1/3+\varepsilon}\right).$$

Our theorems imply that our conjecture (1.9) is true in mean. That is, (1.9) is reasonable in view of our theorems.

Remark 6. The above conjecture (ii) suggests that $\sigma_0+\sigma=3/2$ might be the double analogue of the critical line of the Riemann zeta-function $\Re s=1/2$. On the other hand, in view of the result of Nakamura and Pańkowski mentioned above, we see that another candidate of the double analogue of the critical line is $\sigma_0+\sigma=1$. At present it is not clear which is more plausible.

Remark 7. We cannot expect the analogue of the Riemann hypothesis on the location of zeros. In fact, Theorem 5.1 of Nakamura and Pańkowski [11] asserts (in the double zeta case) that for any $1/2<\sigma_1<\sigma_2<1$, $\zeta_2(s,s)$ has $\asymp T$ non-trivial zeros in the rectangle $\sigma_1<\sigma<\sigma_2$, $0<t<T$.

2. Sketches of the proofs of Theorems 1 and 2

In this section, we explain the proofs of Theorems 1 and 2. First we give a sketch of the proof of Theorem 1.

Let $s_0=\sigma_0+it_0\in\mathbb{C}$ with $\sigma_0>1$ and $s=\sigma+it\in\mathbb{C}$ with $\sigma>1$. Using the definition (1.4), we can see that

$$\int_2^T |\zeta_2(s_0,s)|^2 dt = \zeta_2^{[2]}(s_0,2\sigma)(T-2)$$
$$+\sum_{\substack{m_1,m_2,n_1,n_2\geq 1\\ m_1+n_1\neq m_2+n_2}} \frac{1}{m_1^{s_0}m_2^{\overline{s_0}}(m_1+n_1)^\sigma(m_2+n_2)^\sigma}\int_2^T\left(\frac{m_2+n_2}{m_1+n_1}\right)^{it}dt.$$

Separating the second term on the right-hand side into two parts according to the cases $m_1+n_1<m_2+n_2\leq 2(m_1+n_1)$ and $m_2+n_2>2(m_1+n_1)$, and argue similarly to the proof of [12, Theorem 7.2], we can show that each part has the order $O(1)$ when $\sigma_0>1$ and $\sigma>1$. This implies Theorem 1.

Next we proceed to the proof of Theorem 2. In order to give its proof, it is necessary to prepare the double version of the following well-known result given by Hardy and Littlewood (see [12, Theorem 4.11]): *Let $\sigma_1>0$, $x\geq 1$ and $C>1$. Suppose $s=\sigma+it\in\mathbb{C}$ with $\sigma\geq\sigma_1$ and $|t|\leq 2\pi x/C$. Then*

$$\zeta(s)=\sum_{1\leq n\leq x}\frac{1}{n^s}-\frac{x^{1-s}}{1-s}+O\left(x^{-\sigma}\right)\quad (x\to\infty). \tag{2.1}$$

The double series analogue of (2.1) is as follows.

Theorem 8. *Let $s_0=\sigma_0+it_0\in\mathbb{C}$, $s=\sigma+it\in\mathbb{C}\setminus\{1\}$, $x\geq 1$ and $C>1$. Suppose $\sigma>\max(0,2-\sigma_0)$ and $|t|\leq 2\pi x/C$. Then*

$$\zeta_2(s_0,s)=\sum_{m=1}^\infty\sum_{1\leq n\leq x}\frac{1}{m^{s_0}(m+n)^s}-\frac{1}{1-s}\sum_{m=1}^\infty\frac{1}{m^{s_0}(m+x)^{s-1}}$$
$$+\begin{cases} O(x^{-\sigma}) & (\sigma_0>1)\\ O(x^{-\sigma}\log x) & (\sigma_0=1)\\ O(x^{1-\sigma-\sigma_0}) & (\sigma_0<1)\end{cases}\qquad (x\to\infty). \tag{2.2}$$

In order to prove this theorem, we need the following lemma.

Lemma 9 ([12] Lemma 4.10). *Let $f(x)$ be a real function with a continuous and steadily decreasing derivative $f'(x)$ in (a,b), and let $f'(b)=\alpha$, $f'(a)=\beta$. Let $g(x)$ be a real positive decreasing function with a continuous derivative $g'(x)$, satisfying that $|g'(x)|$ is steadily decreasing. Then*

$$\sum_{a<n\le b} g(n)e^{2\pi i f(n)} = \sum_{\substack{\nu\in\mathbb{Z} \\ \alpha-\eta<\nu<\beta+\eta}} \int_a^b g(x)e^{2\pi i(f(x)-\nu x)}dx + O\left(g(a)\log(\beta-\alpha+2)\right) + O\left(|g'(a)|\right) \tag{2.3}$$

for an arbitrary $\eta\in(0,1)$.

Using this lemma, we can give a proof of Theorem 8.

A sketch of the proof of Theorem 8. First we assume that $\sigma_0>1$ and $\sigma>1$. Then, using the Euler-Maclaurin formula (see [12, Equation (2.1.2)]), we have

$$\begin{aligned}
&\sum_{m=1}^{\infty}\frac{1}{m^{s_0}}\sum_{n=1}^{\infty}\frac{1}{(m+n)^s} \\
&=\sum_{m=1}^{\infty}\frac{1}{m^{s_0}}\sum_{n=1}^{N}\frac{1}{(m+n)^s}-\sum_{m=1}^{\infty}\frac{(m+N)^{1-s}}{m^{s_0}(1-s)} \\
&-s\sum_{m=1}^{\infty}\frac{1}{m^{s_0}}\int_{m+N}^{\infty}\frac{y-[y]-1/2}{y^{s+1}}dy-\frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m^{s_0}(m+N)^s} \\
&=A_1-A_2-A_3-A_4 \text{ (say)},
\end{aligned} \tag{2.4}$$

where we can check that the right-hand side can be continued to the desired region. In fact, the terms A_1 and A_4 are absolutely convergent in the region $\sigma_0+\sigma>1$, and in this region

$$A_4 = O\left(\sum_{m=1}^{\infty}\frac{1}{m^{\sigma_0}(m+N)^{\sigma}}\right). \tag{2.5}$$

The integral in A_3 is absolutely convergent if $\sigma>0$, and is $O(\sigma^{-1}(m+N)^{-\sigma})$. Therefore A_3 can be continued to the region $\sigma>0$, $\sigma_0+\sigma>1$ and

$$A_3 = O\left(\sum_{m=1}^{\infty}\frac{|s|/\sigma}{m^{\sigma_0}(m+N)^{\sigma}}\right) \tag{2.6}$$

there. The term A_2 is absolutely convergent for $\sigma_0+\sigma>2$, $s\ne1$.

Hereafter we assume $N>x$ and fix $m\in\mathbb{N}$. For $\sigma>0$ and a small η, we obtain by Lemma 9 that

$$\begin{aligned}
\sum_{x<n\le N}\frac{1}{(m+n)^s} &= \sum_{x<n\le N}\frac{e^{-it\log(m+n)}}{(m+n)^{\sigma}} = \int_x^N\frac{1}{(m+u)^s}du + O\left((m+x)^{-\sigma}\right) \\
&= \frac{(m+N)^{1-s}-(m+x)^{1-s}}{1-s} + O\left((m+x)^{-\sigma}\right).
\end{aligned} \tag{2.7}$$

In other words, denoting the above error term by $E(s;x,m,N)$, we find that this function is entire in s (the point $s=1$ is a removable singularity) and satisfies

$$E(s;x,m,N)=O\left((m+x)^{-\sigma}\right) \tag{2.8}$$

uniformly in N in the region $\sigma>0$. Therefore we have

$$\begin{aligned}\zeta_2(s_0,s)=&\sum_{m=1}^{\infty}\sum_{n\le x}\frac{1}{m^{s_0}(m+n)^s}-\frac{1}{1-s}\sum_{m=1}^{\infty}\frac{1}{m^{s_0}(m+x)^{s-1}}\\&+\sum_{m=1}^{\infty}\frac{E(s;x,m,N)}{m^{s_0}}-A_3-A_4\end{aligned} \tag{2.9}$$

in the region $\sigma>\max(0,2-\sigma_0)$, $s\ne 1$. Letting $N\to\infty$, we obtain Theorem 8. □

Now we start the proof of Theorem 2.

A sketch of the proof of Theorem 2. Let $s_0=\sigma_0+it_0\in\mathbb{C}$ with $\sigma_0>1$ and $s=\sigma+it\in\mathbb{C}\setminus\{1\}$ with $1/2<\sigma\le 1$, $\sigma_0+\sigma>2$. Setting $C=2\pi$ and $x=t$ in (2.2), we have

$$\zeta_2(s_0,s)=\sum_{m=1}^{\infty}\sum_{1\le n\le t}\frac{1}{m^{s_0}(m+n)^s}+O\left(t^{-\sigma}\right)\quad(t\to\infty). \tag{2.10}$$

We denote the first term on the right-hand side by $\Sigma_1(s_0,s)$. Let $M(n_1,n_2)=\max\{n_1,n_2,2\}$. Then

$$\begin{aligned}&\int_2^T|\Sigma_1(s_0,s)|^2dt\\&=\sum_{m_1\ge1}\sum_{m_2\ge1}\frac{1}{m_1^{s_0}m_2^{\overline{s_0}}}\sum_{n_1\le T}\sum_{\substack{n_2\le T\\ m_1+n_1=m_2+n_2}}\frac{1}{(m_1+n_1)^{2\sigma}}(T-M(n_1,n_2))\\&+\sum_{m_1\ge1}\sum_{m_2\ge1}\frac{1}{m_1^{s_0}m_2^{\overline{s_0}}}\sum_{n_1\le T}\sum_{\substack{n_2\le T\\ m_1+n_1\ne m_2+n_2}}\frac{1}{(m_1+n_1)^{\sigma}(m_2+n_2)^{\sigma}}\\&\times\frac{e^{iT\log((m_2+n_2)/(m_1+n_1))}-e^{iM(n_1,n_2)\log((m_2+n_2)/(m_1+n_1))}}{i\log((m_2+n_2)/(m_1+n_1))}.\end{aligned} \tag{2.11}$$

We denote the first and the second term on the right-hand side by S_1T-S_2 and S_3, respectively. Then we can see that

$$S_1T=\zeta_2^{[2]}(s_0,2\sigma)T+O\left(T^{2-2\sigma}\right), \tag{2.12}$$

and

$$S_2\ll\begin{cases}T^{2-2\sigma} & (1/2<\sigma<1)\\ \log T & (\sigma=1),\end{cases}$$

because $\sigma_0>1$. Also

$$S_3\ll\sum_{m_1,m_2\ge1}\frac{1}{(m_1m_2)^{\sigma_0}}\sum_{\substack{n_1,n_2\le T\\ m_1+n_1<m_2+n_2\le2(m_1+n_1)}}\frac{1}{(m_1+n_1)^{\sigma}(m_2+n_2)^{\sigma}}\frac{1}{\log\frac{m_2+n_2}{m_1+n_1}}$$

$$+\sum_{m_1,m_2\geq 1}\frac{1}{(m_1m_2)^{\sigma_0}}\sum_{\substack{n_1,n_2\leq T\\ m_2+n_2>2(m_1+n_1)}}\frac{1}{(m_1+n_1)^{\sigma}(m_2+n_2)^{\sigma}}\frac{1}{\log\frac{m_2+n_2}{m_1+n_1}}.$$

We denote the first and the second term by W_1 and W_2, respectively. Then we have

$$W_1 \ll \begin{cases} T^{2-2\sigma}\log T & (1/2<\sigma<1) \\ (\log T)^2 & (\sigma=1), \end{cases}$$

$$W_2 \ll \begin{cases} T^{2-2\sigma} & (1/2<\sigma<1) \\ (\log T)^2 & (\sigma=1). \end{cases}$$

Combining these results, we obtain

$$\begin{aligned} &\int_2^T |\zeta_2(s_0,s)|^2 dt \\ &= \int_2^T |\Sigma_1(s_0,s)+O\left(t^{-\sigma}\right)|^2 dt \\ &= \int_2^T |\Sigma_1(s_0,s)|^2 dt + O\left(\int_2^T |\Sigma_1(s_0,s)|\, t^{-\sigma} dt\right) + O\left(\int_2^T t^{-2\sigma} dt\right). \end{aligned} \tag{2.13}$$

By the Cauchy-Schwarz inequality, we can obtain Theorem 2. □

3. A sketch of the proof of Theorem 3

In the previous section, we gave the proof of Theorem 2 by use of (2.10) which comes from Theorem 8. However Theorem 8 holds under the conditions $\sigma_0>0$ and $\sigma_0+\sigma>2$. Hence we cannot use it for $3/2<\sigma_0+\sigma\leq 2$. In order to prove a mean value result in the latter case, we have to prepare another approximate formula for $\zeta_2(s_0,s)$.

Theorem 10. *Let $s_0=\sigma_0+it_0\in\mathbb{C}$ with $0<\sigma_0<3/2$ and $s=\sigma+it\in\mathbb{C}$ with $\sigma>1/2$, $\sigma_0+\sigma>1$, $s\neq 1$, and $s_0+s\neq 2$. Then*

$$\zeta_2(s_0,s)=\sum_{m=1}^{\infty}\sum_{n\leq t}\frac{1}{m^{s_0}(m+n)^s}+\begin{cases} O\left(t^{1-\sigma_0-\sigma}\right) & (\sigma_0<1) \\ O\left(t^{-\sigma}\log t\right) & (\sigma_0=1) \\ O\left(t^{-\sigma}\right) & (\sigma_0>1). \end{cases} \tag{3.1}$$

In order to prove this theorem, we begin with (2.9) with $x=t$. As was discussed in the proof of Theorem 8, all but the second term on the right-hand side of (2.9) are convergent in $\sigma>0, \sigma_0+\sigma>1$, so the remaining task is to study the second term.

First we assume $\sigma_0+\sigma>2$, $s\neq 1$. Then by the Euler-Maclaurin formula we have

$$\begin{aligned} &\frac{1}{1-s}\sum_{m=1}^{\infty}\frac{1}{m^{s_0}(m+t)^{s-1}} \\ &=\frac{1}{1-s}\int_1^{\infty}\frac{dy}{y^{s_0}t^{s-1}\left(1+\frac{y}{t}\right)^{s-1}} \end{aligned}$$

$$+\frac{1}{1-s}\int_1^\infty \left(y-[y]-\frac{1}{2}\right)\left(-\frac{s_0}{y^{s_0+1}(y+t)^{s-1}}+\frac{1-s}{y^{s_0}(y+t)^s}\right)dy$$
$$+\frac{1}{2(1-s)}(1+t)^{1-s}$$
$$=g(s_0,s)+Y_2+Y_3 \quad (say). \tag{3.2}$$

We can find that Y_2+Y_3 can be continued to the region $\sigma_0>0$, $\sigma_0+\sigma>1$ and $s\neq 1$, and in this region satisfies

$$Y_2+Y_3=\begin{cases} O\left(t^{1-\sigma_0-\sigma}\right) & (0<\sigma_0<1;\ \sigma_0+\sigma>1)\\ O\left(t^{-\sigma}\log t\right) & (\sigma_0=1;\ \sigma_0+\sigma>1)\\ O\left(t^{-\sigma}\right) & (\sigma_0>1;\ \sigma_0+\sigma>1).\end{cases} \tag{3.3}$$

As for $g(s_0,s)$, we use the classical Mellin-Barnes integral formula

$$(1+\lambda)^{-s}=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)}\lambda^z dz, \tag{3.4}$$

where s, λ are complex numbers with $\sigma=\Re s>0$, $|\arg\lambda|<\pi$, $\lambda\neq 0$, c is real with $-\sigma<c<0$, and the path (c) of integration is the vertical line $\Re z=c$. We apply (3.4) with $\lambda=y/t$ to $g(s_0,s)$ and shift the path of integration suitably to obtain that $g(s_0,s)$ can be continued meromorphically to the region $\sigma_0<3/2$ and $\sigma>1/2$, and satisfies

$$g(s_0,s)=\begin{cases} O\left(t^{-\sigma}\right) & (s_0\neq 1)\\ O\left(t^{-\sigma}\log t\right) & (s_0=1)\end{cases}$$

in this region, except for the singularities

$$s=1,\quad s_0+s=2,1,0,-1,-2,-3,-4,\ldots. \tag{3.5}$$

From these results, we find that the right-hand side of (3.2) can be continued to the region $\sigma_0<3/2$, $\sigma>1/2$, $\sigma_0+\sigma>1$, and satisfies the estimates proved above. On the other hand, the last three terms on the right-hand side of (2.9) (with $x=t$) are estimated by (2.5), (2.6), and (2.8), respectively. Thus we obtain Theorem 10.

Based on these results, we finally give the proof of Theorem 3.

A sketch of the proof of Theorem 3. We let $s_0\in\mathbb{C}$ with $1/2<\sigma_0<3/2$ and $s\in\mathbb{C}$ with $1/2<\sigma\leq 1$ with $3/2<\sigma_0+\sigma\leq 2$. We further assume that $s_0+s\neq 2$. Formula (2.11) holds also in this region, whose right-hand side is $S_1T-S_2+S_3$. Estimating S_1 and S_2, we can obtain

$$S_1=\zeta_2^{[2]}(s_0,2\sigma)+\begin{cases} O\left(T^{3-2\sigma-2\sigma_0}\right) & (\frac{1}{2}<\sigma_0<1)\\ O\left(T^{1-2\sigma}(\log T)^2\right) & (\sigma_0=1)\\ O\left(T^{1-2\sigma}\right) & (1<\sigma_0<\frac{3}{2}),\end{cases} \tag{3.6}$$

$$S_2=\begin{cases} O\left(T^{4-2\sigma_0-2\sigma}\right) & (\frac{1}{2}<\sigma_0<1,\frac{1}{2}<\sigma\leq 1)\\ O\left(T^{2-2\sigma}(\log T)^2\right) & (\sigma_0=1,\frac{1}{2}<\sigma<1)\\ O\left((\log T)^3\right) & (\sigma_0=1,\sigma=1)\\ O\left(T^{2-2\sigma}\right) & (1<\sigma_0<\frac{3}{2},\frac{1}{2}<\sigma<1),\end{cases} \tag{3.7}$$

where we have to note that $3/2 < \sigma_0 + \sigma < 2$ in the first case, and $\sigma \neq 1$ (because if $\sigma = 1$ then $\sigma_0 + \sigma > 2$) in the fourth case.

As for S_3, it is necessary to estimate it more carefully. Similarly to the argument in the previous section, we have

$$S_3 \ll \sum_{m_1,m_2\geq 1} \frac{1}{(m_1m_2)^{\sigma_0}} \sum_{\substack{n_1,n_2\leq T\\ m_1+n_1<m_2+n_2\leq 2(m_1+n_1)}} \frac{1}{(m_1+n_1)^\sigma(m_2+n_2)^\sigma} \frac{1}{\log\frac{m_2+n_2}{m_1+n_1}}$$
$$+ \sum_{m_1,m_2\geq 1} \frac{1}{(m_1m_2)^{\sigma_0}} \sum_{\substack{n_1,n_2\leq T\\ m_2+n_2>2(m_1+n_1)}} \frac{1}{(m_1+n_1)^\sigma(m_2+n_2)^\sigma} \frac{1}{\log\frac{m_2+n_2}{m_1+n_1}},$$

which we denote by $W_1 + W_2$. We can estimate

$$\begin{aligned} W_2 &\ll \sum_{m_1,m_2\geq 1} \frac{1}{(m_1m_2)^{\sigma_0}} \sum_{\substack{n_1,n_2\leq T\\ m_2+n_2>2(m_1+n_1)}} \frac{1}{(m_1+n_1)^\sigma(m_2+n_2)^\sigma} \\ &= \sum_{\substack{m_1\geq 1\\ n_1\leq T}} \frac{1}{m_1^{\sigma_0}(m_1+n_1)^\sigma} \sum_{k>2(m_1+n_1)} \frac{1}{k^\sigma} \sum_{\substack{m_2\geq 1, n_2\leq T\\ m_2+n_2=k}} \frac{1}{m_2^{\sigma_0}} \\ &= \sum_{\substack{m_1\leq T\\ n_1\leq T}} + \sum_{\substack{m_1>T\\ n_1\leq T}} = W_{21} + W_{22}, \end{aligned}$$

say, and we can estimate

$$W_{22} \ll T^{4-2\sigma_0-2\sigma}, \tag{3.8}$$

$$W_{21} \ll \begin{cases} T^{4-2\sigma_0-2\sigma} & (\frac{1}{2}<\sigma_0<1, \frac{1}{2}<\sigma<1) \\ T^{2-2\sigma_0}\log T & (\frac{1}{2}<\sigma_0<1, \sigma=1) \\ T^{2-2\sigma}(\log T)^2 & (\sigma_0=1, \frac{1}{2}<\sigma<1) \\ (\log T)^4 & (\sigma_0=1, \sigma=1) \\ T^{2-2\sigma} & (1<\sigma_0<\frac{3}{2}, \frac{1}{2}<\sigma<1). \end{cases} \tag{3.9}$$

Similarly we divide W_1 into two parts and estimate each part separately. Consequently we obtain

$$\begin{aligned} S_3 &= W_1 + W_2 \\ &\ll \begin{cases} T^{4-2\sigma_0-2\sigma}\log T & (\frac{1}{2}<\sigma_0<1, \frac{1}{2}<\sigma<1) \\ T^{2-2\sigma_0}(\log T)^2 & (\frac{1}{2}<\sigma_0<1, \sigma=1) \\ T^{2-2\sigma}(\log T)^3 & (\sigma_0=1, \frac{1}{2}<\sigma<1) \\ (\log T)^4 & (\sigma_0=1, \sigma=1) \\ T^{2-2\sigma}\log T & (1<\sigma_0<\frac{3}{2}, \frac{1}{2}<\sigma<1). \end{cases} \end{aligned} \tag{3.10}$$

Thus, combining (3.6), (3.7), (3.10) with Theorem 10, and using the Cauchy-Schwarz inequality, we obtain Theorem 3. □

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K. Matsumoto
Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya 464-8602 Japan
E-mail: kohjimat@math.nagoya-u.ac.jp

H. Tsumura
Department of Mathematics and Information Sciences, Tokyo Metropolitan University,
1-1, Minami-Ohsawa, Hachioji, Tokyo 192-0397 Japan
E-mail: tsumura@tmu.ac.jp